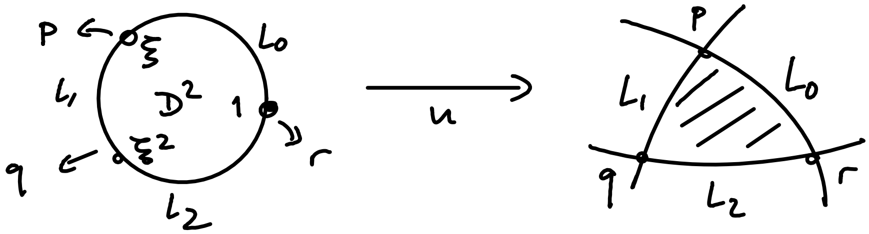
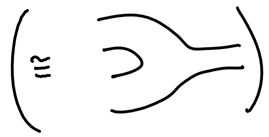
R. Vianna – 10/2/11 – Lagr. Floer homology (continued)

Product structure: $CF^*(L_0,L_1)\otimes CF^*(L_1,L_2)\longrightarrow CF^*(L_0,L_2)$

Look at





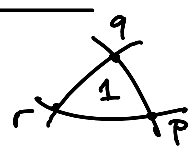
$$p\cdot q=\sum_{\substack{r\in L_0\cap L_2\\ [u]\in\pi_2/\,\mathrm{ind}[u]=0}}\#\mathcal{M}(p,q,r,[u],J)\,T^{\omega(u)}\,r$$

signed count, assuming transversality & orientability

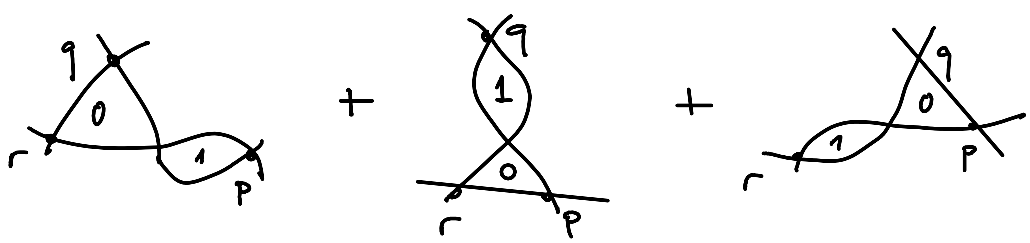
If no bubbling, then $\mathcal{M}$ can be compactified by adding broken trajectories.

Prop: If $[\omega].\pi_2(M,L_i)=0$, then the product is well-defined, and satisfies (signed) Leibniz rule; the induced product on HF^* is associative.

Idea for Leibniz rule: the boundary of an index 1 (1-dim'l) moduli space of



consists of broken configs.



⇒ signed count of these vanishes, i.e.

$$\pm(\partial p)\cdot q\pm p\cdot(\partial q)+\partial(p\cdot q)=0\quad\blacktriangle$$

Grading: defining a $\mathbb{Z}$-grading on Floer homology is possible when $c_1(M)=0$ and $\mu_L=0$.

Recall: the Lagrangian Grassmannian has $\pi_1 = \mathbb{Z}$, & has universal cover $\widetilde{LGr}_n$.

$c_1(TM)$ is the obstruction to lifting the bundle of Lag. planes $\begin{array}{c} LGr(TM) \\ \downarrow \\ M \end{array}$ to a $\widetilde{LGr}$-bundle $\widetilde{LGr}(TM)$.

Namely, assume $c_1(TM) = 0 \Rightarrow c_1(\Lambda^n T^*M^{1,0}) = c_1(T^*M^{1,0}) = 0$

$\Leftrightarrow \exists$ global nonvanishing section $\Omega \in \Omega^{n,0}(M)$.

- Given a Lag. plane $\ell \subseteq T_xM$, can pick a frame $(e_1, \dots, e_n)$ of ℓ st. $(e_1, Je_1, \dots, e_n, Je_n)$ sympl. basis. Writing $dz_i = e_i^* + i(Je_i)^*$, $\Omega_{|x} = f\, dz_1 \wedge \dots \wedge dz_n$, $f \in \mathbb{C}^*$

  $\rightarrow$ $\Omega_{|\ell}$ is automatically nonzero.

  Picking a real volume element vol_ℓ, $\Omega_{|\ell} = \alpha\, vol_\ell$ $\quad \alpha \in \mathbb{C}^*$

  $\rightsquigarrow \arg(\ell) := \arg(\alpha) \in \mathbb{R}/\pi\mathbb{Z}$ (only π if orientation on ℓ not fixed in advance).

Now, define $\widetilde{LGr}(T_xM) := \left\{ (\ell, \theta) \;\middle|\; \begin{array}{l} \ell \in LGr(T_xM),\ \theta \in \mathbb{R}, \\ \theta \equiv \arg(\Omega_{|\ell}) \mod \pi \end{array} \right\}$

- Now, want to lift

$$\begin{tikzcd} & \widetilde{LGr}(TM) \arrow[d] \\ & LGr(TM) \arrow[d] \\ L \arrow[r, hook] \arrow[ur, "TL"] \arrow[uur, dashed, "?"] & M \end{tikzcd}$$

ie. find a real lift of $\arg(\Omega_{|TL}) : L \to S^1$

The obstruction to this is $\mu_L = [L \xrightarrow{\arg} S^1] \in [L, S^1] \cong H^1(L, \mathbb{Z})$.

Namely, $\forall \gamma \in \pi_1(L)$, $\langle \mu_L, \gamma \rangle$ = homotopy class of $\left(\arg \Omega_{|T_{\gamma(t)}L}\right) \in \pi_1(S^1) \in \mathbb{Z}$

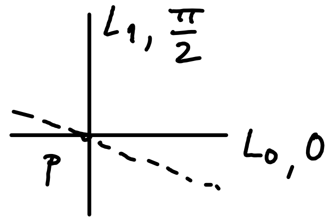
$\equiv$ Maslov index of the loop γ

<u>NB</u>: $c_1(M) = 0 \Rightarrow$ Maslov index of a class $[u] \in \pi_2(M, L)$ only depends on its boundary loop $\in \pi_1(L)$

If $\mu_L = 0$ then can lift L to a "graded Lagrangian".

Then we can assign a degree to an intersection $p \in L_0 \cap L_1$

Ex:



rotate TL_0 slightly clockwise $\rightsquigarrow (T_p L_0)^-$

Then $\deg(p) :=$ Maslov index of path from $(T_p L_0)^-$ to $T_p L_1$ relative to $T_p L_0$ inside $\widetilde{LGr}$.

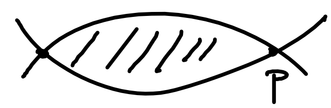
In dim. 1, $\deg(p) = \left\lceil \frac{\theta_1 - \theta_0}{\pi} \right\rceil$

Twisted coefficients: (E_i, ∇_i) vector bundles with flat ∇_i

$$\begin{matrix} (E_i, \nabla_i) \\ \downarrow \\ L_i \end{matrix}$$

$$\rightsquigarrow CF^*((L_0, E_0, \nabla_0), (L_1, E_1, \nabla_1)) := \bigoplus_{p \in L_0 \cap L_1} \operatorname{Hom}(E_{0p}, E_{1p}) \otimes \Lambda$$

Differential still counts holom. strips



$w_1 : E_{0p} \to E_{1p}$

$$\rightsquigarrow \partial w_1 = \sum_{\substack{q \in L_0 \cap L_1 \\ [u] / \operatorname{ind}[u] = 1}} \# \mathcal{M}(p, q, [u], J)\, T^{\omega(u)} \cdot \underbrace{w_{[u]}}_{\in\, \operatorname{Hom}(E_{0q}, E_{1q})}$$

where $w_{[u]} : E_{0q} \xrightarrow[\text{parallel transport along boundary wrt } \nabla_0]{} E_{0p} \xrightarrow{w_1} E_{1p} \xrightarrow[\text{parallel transport for } \nabla_1 \text{ along } \partial \text{ of disc}]{} E_{1q}$

(only depends on homotopy class of ∂u).

Of special interest to us: $E_j \cong \mathbb{C} \times L_j$ trivial line bundle

$\nabla_j =$ flat $U(1)$ connection, $\nabla_j = d + iA_j$, $A_j \in \Omega^1(L_j, \mathbb{R})$, $dA_j = 0$

Then $CF = \bigoplus_{p \in L_0 \cap L_1} \overset{\Lambda \otimes \mathbb{C}}{\overset{\shortparallel}{\Lambda_{\mathbb{C}}}}\, P$, $\quad P : E_{0p} \cong \mathbb{C} \xrightarrow{id} \mathbb{C} \cong E_{1p}$

and now $\partial p = \sum_{q,[u]} \#\mathcal{M}(p,q,[u],J)\, T^{\omega(u)} \mathrm{hol}(\partial u)\, q$.

$$\mathrm{hol}(\partial u) = \exp\Big(i \int_{\partial u} A_j\Big) \in S^1$$

$\hookrightarrow A_0, A_1$ depending on which part of boundary.

<u>Fukaya A_∞(pre) category</u>:

- objects $= \mathcal{L} = (L, E, \nabla)$ L compact, spin, doesn't bound holom. discs
  (E,∇) flat bundle
- morphisms: for $\mathcal{L}_0 \pitchfork \mathcal{L}_1$, $\mathrm{Hom}(\mathcal{L}_0,\mathcal{L}_1) = CF^*(\mathcal{L}_0,\mathcal{L}_1)$
- differential, products, higher products = twisted Floer theory.

$\star$ Physicists think of T as an actual parameter, not a formal variable.
(& hope for convergence of the power series).

$T \longleftrightarrow e^{-2\pi t}$ $(\leftrightarrow$ sympl. geom. on $(M, \omega_t = t\cdot\omega))$

$t \to \infty$ large volume limit.